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Physics 111 – Seminar 3

Response of an Oscillator to an Impulse and Green's Method

Green's Method can be used to solve **linear inhomogeneous** differential equations, such as the equation describing the motion of a damped oscillator:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F(t)}{m}. \tag{1}$$

Since this equation is linear, it obeys the **principle of superposition**: if we have two solutions, $x_1(t)$ and $x_2(t)$, to two different forcing functions, $F_1(t)$ and $F_2(t)$, then $x_1(t) + x_2(t)$ is a solution to the forcing function $F_1(t) + F_2(t)$.

We can always write $F(t)$ as a Fourier series, but then our solution is in the form of an infinite sum. Green discovered a clever way to express $F(t)$ in terms of the delta function that results in a nice analytic solution.

Green's Function, $G(t,t')$, is defined *as the solution to the ODE when the forcing function is equal to the* ***delta function***. In the case of the damped oscillator,

$$\ddot{G} + 2\beta\dot{G} + \omega_0^2 G = \frac{\delta(t - t')}{m}.$$

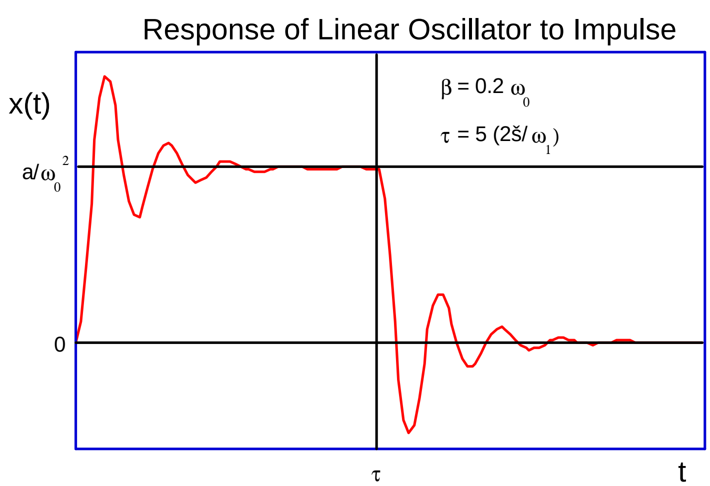
To determine G, first consider the exact response of the system to the Heaviside step function,

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = a, \qquad t > t_0$$

in which a is acceleration. The homogeneous solution (set $a = 0$) is a cosine and sine sum with arbitrary constants, and the inhomogeneous solution is just a constant, a/ω_0^2. The constants can be found by applying initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$, giving the solution:

$$x(t) = \frac{H(t_0)}{\omega_0^2}\left[1 - e^{-\beta(t-t_0)}\cos\omega_1(t - t_0) - \frac{\beta e^{-\beta(t-t_0)}}{\omega_1}\sin\omega_1(t - t_0)\right]$$

By subtracting a step function at $t = t$, we get an impulse function of width t and height a.



If we keep the product at constant as we let $t \to 0$, we get the response to a delta function (see M&T p. 143-144):

$$x(t) = \frac{\tau \cdot a(t)}{\omega_1} e^{-\beta(t-t_0)} \sin \omega_1 (t - t_0), \quad t > t_0$$

We can express any forcing function as the sum of impulse functions. If we define $F_n(t)$ as $a_n(t) \cdot m$ when $t_n < t < t_{n+1}$, where $t = t_{n+1} - t_n$, we can write

$$\frac{F(t)}{m} = \lim_{\tau \to 0} \sum_{n=-\infty}^{\infty} \frac{F_n(t)}{m} = \int_{-\infty}^{t} \frac{F(t')}{m} \delta(t - t') dt'$$

We know the solution for the *n*th impulse (it acts over interval $t = t_{n+1} - t_n$), so we can sum these to find the solution to the original forcing function:

$$x(t) = \sum_{n=-\infty}^{t} x_n(t) = \sum_{n=-\infty}^{t} \frac{\tau \cdot a_n(t_n)}{\omega_1} e^{-\beta(t-t_0)} \sin \omega_1 (t - t_n), \qquad t_N < t < t_{N+1}.$$

Taking the limit as $t \to 0$ and writing t_n as t', we get the integral

$$x(t) = \int_{-\infty}^{t} \frac{a(t')}{\omega_1} e^{-\beta(t-t')} \sin \omega_1 (t - t') dt',$$

and when we say that

$$G(t,t') = \begin{cases} \dfrac{1}{m\omega_1} e^{-\beta(t-t')} \sin \omega_1 (t - t'), & t \geq t' \\ 0, & t < t' \end{cases} \qquad (2)$$

we have

$$x(t) = \int_{-\infty}^{t} F(t') G(t,t') dt'. \qquad (3)$$

Note that while Equation 3 is always true, Equation 2 is specific to this problem: a linear oscillator initially at rest at equilibrium. The *Green's function contains the initial conditions*, so putting any forcing function into Equation 3 will give you an exact solution to the problem.

Though it may be difficult to calculate the integral (as you will see in problem 3-42), Green's Method always gives you a solution.