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Physics 115 - Seminar 4

The Born Approximation (Time-Dependent Description)

We consider our incoming waves as plane waves, $|\vec{p}_i\rangle$, and we want to evalulate the scattering S matrix, which shows what happens to our incoming waves after scattering. The approximation is that the scattering potential V can be treated as a perturbation, allowing us to use Fermi's Golden Rule:

$$\begin{aligned} R_{i\to d\Omega} &= \frac{2\pi}{\hbar}\left[\int_0^\infty \left|\langle\vec{p}_f\,|V|\,\vec{p}_i\rangle\right|^2 \delta\left(\frac{p_f^2}{2\mu}-\frac{p_i^2}{2\mu}\right)p_f^2 dp_f\right]d\Omega \\ &= \frac{2\pi}{\hbar}\left[\int_0^\infty \left|\langle\vec{p}_f\,|V|\,\vec{p}_i\rangle\right|^2 2\mu\delta\left(p_f^2-p_i^2\right)p_f^2\frac{dp_f^2}{2p_f}\right]d\Omega \\ &= \frac{2\pi}{\hbar}\left|\langle\vec{p}_f\,|V|\,\vec{p}_i\rangle\right|^2 \mu p_i d\Omega. \end{aligned} \tag{1}$$

The incoming probability rate is

$$j_{\text{inc}} = \frac{\hbar k}{\mu}\left(\frac{1}{2\pi\hbar}\right)^3 = \frac{p_i}{\mu}\left(\frac{1}{2\pi\hbar}\right)^3 \tag{2}$$

in the $\vec{p}_i$ direction, so

$$\frac{d\sigma}{d\Omega}d\Omega = \frac{R_{i\to d\Omega}}{j_{\text{inc}}} = (2\pi)^4\mu^2\hbar^2\left|\langle\vec{p}_f\,|V|\,\vec{p}_i\rangle\right|^2 d\Omega. \tag{3}$$

We can now obtain the differential cross section:

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \left|(2\pi)^2\mu\hbar\int\langle\vec{p}_f|\,\vec{r}\rangle\,V(\vec{r})\,\langle\vec{r}\,|\vec{p}_i\,\rangle\,d^3\vec{r}\right|^2 \\ &= \left|\frac{(2\pi)^2\mu\hbar}{(2\pi\hbar)^3}\int e^{-i\vec{p}_f\cdot\vec{r}/\hbar}V(\vec{r})e^{i\vec{p}_i\cdot\vec{r}/\hbar}d^3\vec{r}\right|^2 \\ &= \left|\frac{\mu}{2\pi\hbar^2}\int e^{-i\vec{q}\cdot\vec{r}}V(\vec{r})d^3\vec{r}\right|^2 \end{aligned} \tag{4}$$

where $\hbar\vec{q} = \vec{p}_f - \vec{p}_i$. As derived in Shankar, $q = 2k\sin(\theta/2)$. We can now identify $f(\theta,\phi)$ as

$$\begin{aligned} f(\theta,\phi) &= -\frac{\mu}{2\pi\hbar^2}\int e^{-i\vec{q}\cdot\vec{r}'}V(\vec{r}')d^3\vec{r}' \\ &= -\frac{2\mu}{\hbar^2}\int\frac{\sin qr'}{q}V(r')r'dr'. \end{aligned} \tag{5}$$

Note that this is the same as the result obtained in the time-independent description of the Born approximation; in fact, the two descriptions agree to all orders.