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Physics 111

Euler's Equations

In most cases, the problem of describing the motion of a rigid body can be separated into two problems, one involving translation and one involving rotation. We describe translation with respect to the inertial *fixed frame*, and rotation with respect to the *body frame*. It is convenient to choose the coordinates of the center of mass to describe motion in the fixed frame, and to use *Eulerian angles* to describe motion in the body frame, in which the origin is the center of mass of the body.

Euler's equations describe the rotation of a body (in the body frame). To derive them, we consider the torque $\mathbf{N}$ in an inertial reference frame:

$$\left(\frac{d\mathbf{L}}{dt}\right)_{fixed} = \mathbf{r}\times\left(\frac{d\mathbf{p}}{dt}\right)_{fixed} = \mathbf{N}$$

Using the general relation between the time derivative of a vector in the fixed and rotating frames, we have

$$\left(\frac{d\mathbf{L}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{L} = \mathbf{N}$$

The first component of this vector equation (the component along the x_1-axis is

$$\dot{L}_1 + (\omega_2 L_3 - \omega_3 L_2) = N_1$$

and if we choose the x_i-axes to be the principle axes, we can use the relation $L_i = I_i\omega_i$ to write all three components as

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\,\omega_2\omega_3 &= N_1 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\,\omega_3\omega_1 &= N_2 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\,\omega_1\omega_2 &= N_3 \end{aligned}$$

These are Euler's equations, and they can be compactly rewritten as

$$(I_i - I_j)\,\omega_i\omega_j - \sum_k (I_k\dot{\omega}_k - N_k)\,\epsilon_{ijk} = 0$$

For force free motion, $\mathbf{N} = 0$, and we have

$$(I_i - I_j)\,\omega_i\omega_j - \sum_k I_k\dot{\omega}_k\epsilon_{ijk} = 0$$

Euler's equations give us the motion of a body with respect to the body frame, but they tell us nothing about the motion of the body frame itself. In gaining the convenience of a constant inertia tensor, we lost knowledge of the orientation of the body frame's axes. Euler's equations cannot be directly integrated to determine the angles specifying this orientation. Instead, we can express the ω_i in terms of the Eulerian angles ϕ, θ, and ψ, resulting in complicated (but solvable) coupled differential equations.