Lisa Larrimore
Physics 115 - Seminar 3

# Field Quantization

Our general approach for quantizing the electromagnetic field is to find classical conjugate coordinates and their conoical commutation relation. By one of Shankar's postulates of quantum mechanics (see Section 4.1), the quantum hamiltonian, $\hat{H}$, is simply the classical hamiltonian, $\mathcal{H}$, with the conjugate coordinates transformed to quantum operators.

We start with the four coordinates describing the potential, $(\phi(\vec{r}), \vec{A}(\vec{r}))$, where we impose the Coulomb gauge so that $\nabla \cdot \vec{A} = 0$ and $\phi = 0$. To find the momenta conjugate to these coordinates, we must find the Lagrangian. Remember that the energy of a field is given by

$$u = \frac{1}{8\pi} \int \left|\vec{E}\right|^2 + \left|\vec{B}\right|^2 d^3\vec{r}. \tag{1}$$

Since $\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$ is a time derivative, like a kinetic energy, and $\vec{B} = \nabla \times \vec{A}$ is a spatial derivative, like a potential energy, we guess at the form of the Lagrangian:

$$\begin{aligned} \mathcal{L} &= \frac{1}{8\pi} \int \left|\vec{E}\right|^2 - \left|\vec{B}\right|^2 d^3\vec{r} \\ &= \frac{1}{8\pi} \int \left|-\frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right|^2 - \left|\nabla \times \vec{A}\right|^2 d^3\vec{r}, \end{aligned} \tag{2}$$

where we have imposed the Coulomb gauge to eliminate $\phi$. We can now find the momentum conjugate to $\vec{A}$, which we will label $\vec{\Pi}$:

$$\vec{\Pi} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{A}}} = \frac{\dot{\vec{A}}}{4\pi c^2} = \frac{\vec{E}}{4\pi c}. \tag{3}$$

By writing Eq. (1) in terms of these conjugate coordinates, the classical Hamiltonian becomes

$$\mathcal{H} = \frac{1}{8\pi} \int \left|-4\pi c\vec{\Pi}\right|^2 + \left|\nabla \times \vec{A}\right|^2 d^3\vec{r}. \tag{4}$$

We cannot simply quantize from here because there are additional constraints on $\vec{\Pi}$ and $\vec{A}$ that render them noncanonical. Gauss's law, $\nabla \cdot \vec{E} = 0$, implies that $\nabla \cdot \vec{\Pi} = 0$, and we still have the Coulomb gauge constraint that $\nabla \cdot \vec{A} = 0$. We thus search for a smaller set of variables that have these constraints built in.

We Fourier transform $\vec{A}$ and $\vec{\Pi}$ to relate their six real components to three complex components of $\vec{a}$ in $\vec{k}$ space:

$$\vec{A}(\vec{r}) = \sum_{\lambda=1}^{3} \int \left(\frac{c^2}{4\pi^2\omega}\right)^{\frac{1}{2}} \left[a(\vec{k}\lambda)\epsilon(\vec{k}\lambda)e^{i\vec{k}\cdot\vec{r}} + a^*(\vec{k}\lambda)\epsilon(\vec{k}\lambda)e^{-i\vec{k}\cdot\vec{r}}\right] d^3\vec{k}, \tag{5}$$

$$\vec{\Pi}(\vec{r}) = \sum_{\lambda=1}^{3} \int \frac{1}{i} \left(\frac{\omega}{64\pi^4c^2}\right)^{\frac{1}{2}} \left[a(\vec{k}\lambda)\epsilon(\vec{k}\lambda)e^{i\vec{k}\cdot\vec{r}} - a^*(\vec{k}\lambda)\epsilon(\vec{k}\lambda)e^{-i\vec{k}\cdot\vec{r}}\right] d^3\vec{k}. \tag{6}$$

Here, $\epsilon(\vec{k}i)$ is perpendicular to $\vec{k}$ for $i = 1, 2$ and parallel to $\vec{k}$ for $i = 3$. The two constraints can now be combined into the condition that $a(\vec{k}3) = 0$, which changes the summation indicies to $\lambda = 1, 2$. We can now write the classical hamiltonian as

$$\mathcal{H} = \sum_{\lambda=1}^{2} \int \omega \left[ a^*(\vec{k}\lambda) a(\vec{k}\lambda) \right] d^3\vec{k}. \tag{7}$$

This is equivalent to a bunch of oscillators with coordinates $p$ and $q$, which we turn into quantum operators $\hat{Q}$ and $\hat{P}$ obeying

$$\left[ \hat{Q}(\vec{k}\lambda), \hat{P}(\vec{k}'\lambda') \right] = i\hbar\{q, p\} = i\hbar\delta_{\lambda,\lambda'}\delta^3\left(\vec{k} - \vec{k}'\right). \tag{8}$$

We then change variables to the raising and lowering operators, $\hat{a}^\dagger$ and $\hat{a}$, in terms of which our quantum hamiltonian is

$$\hat{H} = \sum_{\lambda} \int \left[ \hat{a}^\dagger(\vec{k}\lambda)\hat{a}(\vec{k}\lambda) + \frac{1}{2} \right] \hbar\omega d^3\vec{k}. \tag{9}$$

Note that $\hat{a}^\dagger$ and $\hat{a}$ correspond to the classical variables $a^*/\hbar^{1/2}$ and $a/\hbar^{1/2}$. Also note that in this context, $\hat{a}^\dagger$ and $\hat{a}$ represent the creation and annihilation of photons. Acting on the field ground state $|0\rangle$ with the lowering operator gives 0, but

$$\hat{a}^\dagger(\vec{k}\lambda) |0\rangle = \left| \vec{k}\lambda \right\rangle, \tag{10}$$

where $\left| \vec{k}\lambda \right\rangle$ is a state with momentum $\hbar\vec{k}$ and energy $\hbar\omega$ ... i.e., a photon.