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Physics 115 - Seminar 12

Multimode and Continuous-Mode Quantum Optics

Though dealing with single-modes is computationally and theoretically easier, no real light beam contains photons of only one frequency. In Chapter 6, Loudon shows how single-mode quantum optics can be extended to multimode and continuous-mode descriptions. Multimode optical theory is used to describe a discrete set of allowed modes, such as in an optical cavity of length L where the one-dimensional mode spacing is $\Delta\omega = 2\pi c/L$. Most optical experiments, however, do not have an identifiable cavity; the beam simply goes from source to detector. Without a cavity, $L \to \infty$, which means the allowed modes become continuous.

To describe multiple modes, each Fock state is subscripted, $|\{n_{\mathbf{k}\lambda}\}\rangle$, and these states form a complete set. We will consider wavefronts of a single polarization propagating in the z direction, so the only necessary index is k. We will label the continuous mode states by ω rather than the wavevector.

The destruction and creation operators become $\hat{a}_k$ and $\hat{a}_k^\dagger$ in the discrete case and $\hat{a}(\omega)$ and $\hat{a}^\dagger(\omega)$ in the continuous case. They obey the commutation relations

$$\left[\hat{a}_k, \hat{a}_{k'}^\dagger\right] = \delta_{k,k'}, \quad \left[\hat{a}(\omega), \hat{a}^\dagger(\omega')\right] = \delta(\omega - \omega'), \tag{1}$$

so they commute unless they describe the same mode. Note that $\hat{a}(\omega)\hat{a}^\dagger(\omega)$ is undefined unless it is inside an integral.

Using the multimode formalism, we can describe the coherent and chaotic light that was treated in Chapter 5. The multimode coherent state is given by

$$|\{\alpha\}\rangle = |\alpha_{k_1}\rangle\, |\alpha_{k_2}\rangle\, |\alpha_{k_3}\rangle \cdots, \tag{2}$$

and using the fact that $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$, we see that it is an eigenstate of the positive frequency part of the electric-field operator developed in Chapter 4. This means that multimode coherent light is second-order coherent, as were the classical stable waves described in Chapter 3. Loudon also shows that the classical and quantum theories give the same first- and second-order degrees of coherence for chaotic light.

To describe nonclassical effects, Loudon first develops the continuous-mode formalism, most of which follows straightforwardly from the discrete case. For instance, the positive-frequency electric field operator is given by

$$\hat{E}_T^+(z,t) = i\int_0^\infty d\omega \left(\frac{\hbar\omega}{4\pi\epsilon_0 cA}\right)^{1/2} \hat{a}(\omega)\exp\left[-i\omega\left(t - \frac{z}{c}\right)\right]. \tag{3}$$

One important tool is the number operator,

$$\hat{n} = \int_0^\infty d\omega \hat{a}^\dagger(\omega)\hat{a}(\omega). \tag{4}$$

For many experiments, the frequency bandwith is much smaller than the central frequency, which means that we can extend the lower limit of integration to $-\infty$ without significant error. Then, writing Fourier-transformed operators as

$$\hat{a}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty d\omega \hat{(a)}(\omega)e^{i\omega t}, \tag{5}$$

the number operator is

$$\hat{n} = \int dt \hat{a}^{\dagger}(t)\hat{a}(t) = \int dt \hat{f}(t). \tag{6}$$

This last expression, $\hat{f}(t)$, is the photon flux operator, and its expectation value is the mean photon flux, $f(t)$.

For a stationary light beam (one whose fluctuations do not change with time), which can be experimentally approximated when the duration of the beam is much greater than the time scales of the system, the mean photon flux is, as expected, time-independent:

$$f(t) = \frac{1}{2\pi} \int d\omega \int d\omega' \left\langle \hat{a}^{\dagger}(\omega)\hat{a}(\omega') \right\rangle e^{i(\omega-\omega')t} = \int d\omega \int d\omega' f(\omega)\delta(\omega-\omega')e^{i(\omega-\omega')t} = \int d\omega f(\omega) \equiv F. \tag{7}$$

Similarly, the degrees of coherence for stationary light can be written as

$$g^{(1)}(\tau) = \frac{\left\langle \hat{a}^{\dagger}(t)\hat{a}(t+\tau) \right\rangle}{\left\langle \hat{a}^{\dagger}(t)\hat{(a)}(t) \right\rangle}, \tag{8}$$

and

$$g^{(2)}(\tau) = \frac{\left\langle \hat{a}^{\dagger}(t)\hat{a}\dagger(t+\tau)\hat{a}(t+\tau)\hat{a}(t) \right\rangle}{\langle \hat{a}^{\dagger}(t)\hat{a}(t) \rangle^{2}}, \tag{9}$$

where $\tau = t_2 - t_1 - (z_2 - z_1)/c$.

The theory of continuous-mode quantum optics developed in Section 6.2 is used for the remainder of the chapter, as will be seen in Abram's presentation and the reading for next week.