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Physics 112

Relativistic Electromagnetic Field Transformations

Derivation of Transformation Equations

Since the general transformation rules for electromagnetic fields should be the same no matter how the fields were produced, both Griffiths and Purcell derive these rules based on the simple case of two parallel uniformly charged sheets. Using our text's notation, let $\mathcal{S}_0$ be the system in which a capacitor with plates parallel to the x_0-z_0 plane is at rest. $\mathcal{S}$ is the system moving with velocity $v_0\hat{\mathbf{x}}$ relative to $\mathcal{S}_0$, and $\bar{\mathcal{S}}$ is the system moving with velocity $\bar{v}\hat{\mathbf{x}}$ relative to $\mathcal{S}_0$ and with $v\hat{\mathbf{x}}$ relative to $\mathcal{S}$. To write $\bar{v}$ in terms of v and v_0, you must remember *Einstein's velocity addition rule* (see Example 12.6):

$$\bar{v} = \frac{v + v_0}{1 + vv_0/c^2}. \tag{1}$$

Also remember the definition of γ,

$$\gamma = \frac{1}{(1 - v^2/c^2)^{1/2}}, \tag{2}$$

and the Lorentz contraction rule, which tells us that the length in the x-direction of the moving sheets is contracted

$$l_0 = \gamma_0 l = \bar{\gamma}\bar{l}.$$

Thus, since *charge is invarient*, and since the width in the z-direction is not changed by motion in the x-direction,

$$\frac{\sigma_0}{l_0 w_0} = \frac{\sigma}{l w_0} = \frac{\bar{\sigma}}{\bar{l} w_0} \Rightarrow \sigma_0 = \frac{\sigma}{\gamma_0} = \frac{\bar{\sigma}}{\bar{\gamma}}. \tag{3}$$

We have two sheets with surface charge $\pm\sigma$ and surface current $\mathbf{K}_\pm = \mp\sigma v_{rel}\hat{\mathbf{x}}$. The fields should be familiar - we derived the E-field in Example 2.4 (using Gauss's Law) and the B-field in Example 5.8 (using Ampère's Law):

$$E_y = \frac{\sigma}{\epsilon_0}, \quad B_z = -\mu_0\sigma v_0,$$

$$\bar{E}_y = \frac{\bar{\sigma}}{\epsilon_0}, \quad \bar{B}_z = -\mu_0\bar{\sigma}\bar{v}. \tag{4}$$

Using our charge density transformations from Eq. (3), we can write Eq. (4) as

$$\bar{E}_y = \left(\frac{\bar{\gamma}}{\gamma_0}\right)\frac{\sigma}{\epsilon_0}, \quad \bar{B}_z = -\left(\frac{\bar{\gamma}}{\gamma_0}\right)\mu_0\sigma\bar{v}. \tag{5}$$

From the definition of γ in Eq. (2), the velocity addition rule in Eq. (1), and some algebra, we have

$$\frac{\bar{\gamma}}{\gamma_0} = \left(\frac{1 - v_0^2/c^2}{1 - \bar{v}^2/c^2}\right)^{1/2} = \left(\frac{c^2 - v_0^2}{c^2 - \left(\frac{v+v_0}{1+vv_0/c^2}\right)^2}\right)^{1/2} = \frac{1 + vv_0/c^2}{(1 - v^2/c^2)^{1/2}} = \gamma\left(1 + \frac{vv_0}{c^2}\right).$$

Substituting into Eq. (5), this gives us

$$\bar{E}_y = \gamma\left(1 + \frac{vv_0}{c^2}\right)\frac{\sigma}{\epsilon_0} = \gamma\left(E_y - \frac{v}{c^2\epsilon_0\mu_0}B_z\right) = \gamma\left(E_y - vB_z\right)$$

and

$$\bar{B}_z = -\gamma\left(1 + \frac{vv_0}{c^2}\right)\mu_0\sigma\left(\frac{v + v_0}{1 + vv_0/c^2}\right) = \gamma\left(B_z - \mu_0\epsilon_0 vE_y\right) = \gamma\left(B_z - \frac{v}{c^2}E_y\right).$$

We can similarly see how E_z and B_y transform by aligning the plates parellel to the x-y plane, in which case the charge densities are increased in the same way and the fields in $\mathcal{S}$ become

$$E_z = \frac{\sigma}{\epsilon_0}, \quad B_y = \mu_0\sigma v_0.$$

Since E_z equals our old E_y and B_y equals minus our old B_z, we can write down the transformation rules with these substitutions.

If the capacitor is oriented parallel to the y-z plane, then the contracted dimension is the distance between the capacitors, so the charge density remains the same in all reference frames. The electric field is the same in all frames, so $\bar{E}_x = E_x$. This situation, however, does not tell us the transformation rule for B_x, since there is no magnetic field in any frame. For this, we must construct an entirely different configuration: a solenoid with its center along the x-axis at rest in $\mathcal{S}$. The magnetic field in $\mathcal{S}$ is

$$B_x = \mu_0 n I.$$

In $\bar{\mathcal{S}}$, both n and I change because *length contracts*, increasing the number of turns per length by a factor of γ, and *time dilates*, decreasing the charge per unit time by a factor of $1/\gamma$. Thus,

$$\bar{B}_x = \mu_0 \bar{n} \bar{I} = \mu_0 (\gamma n) \left(\frac{I}{\gamma} \right) = \mu_0 n I = B_x.$$

Our complete transformation rules are now

$$\begin{aligned} \bar{E}_x &= E_x, \quad \bar{E}_y = \gamma(E_y - vB_z), \quad \bar{E}_z = \gamma(E_z + vB_y), \\ \bar{B}_x &= B_x, \quad \bar{B}_y = \gamma\left(B_y + \tfrac{v}{c^2}E_z\right), \quad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right). \end{aligned} \tag{6}$$

Note that the components of both $\mathbf{E}$ and $\mathbf{B}$ parallel to the motion remain unchanged. Also note that if there is *any frame* in which either $\mathbf{E}$ or $\mathbf{B}$ is zero, we have simple relations for the fields in any other system: $\mathbf{B} = 0$ in $\mathcal{S}$ implies that

$$\bar{\mathbf{B}} = \gamma \frac{v}{c^2}(E_x \hat{\mathbf{y}} - E_y \hat{\mathbf{z}}) = \frac{v}{c^2}(\bar{E}_z \hat{\mathbf{y}} - \bar{E}_z \hat{\mathbf{z}}) = -\frac{1}{c^2}(\mathbf{v} \times \bar{\mathbf{E}}), \tag{7}$$

and $\mathbf{E} = 0$ in $\mathcal{S}$ implies that

$$\bar{\mathbf{E}} = -\gamma v(B_z \hat{\mathbf{y}} - B_y \hat{\mathbf{z}}) = -v(\bar{B}_z \hat{\mathbf{y}} - \bar{B}_y \hat{\mathbf{z}}) = \mathbf{v} \times \bar{\mathbf{B}}. \tag{8}$$

Fields of a Point Charge in Uniform Motion (Revisited)

In Example 10.4 Griffiths calculated the electric and magnetic fields of a point charge moving with constant velocity by using the retarded potentials. This problem becomes much easier with our new tools for describing field transformations, as seen in Examples 12.13 and 12.14.

We can write the electric field of a point charge q at rest as

$$\mathbf{E}_0 = \frac{q}{4\pi\epsilon_0} \frac{x_0 \hat{\mathbf{x}} + y_0 \hat{\mathbf{y}} + z_0 \hat{\mathbf{z}}}{(x_0^2 + y_0^2 + z_0^2)^{3/2}}.$$

Since the magnetic field in this frame is zero, we can use the transformaion rules to write E in a frame moving to the right at speed v_0 relative to the rest frame:

$$\mathbf{E}_0 = \frac{q}{4\pi\epsilon_0} \frac{x_0 \hat{\mathbf{x}} + \gamma_0 y_0 \hat{\mathbf{y}} + \gamma_0 z_0 \hat{\mathbf{z}}}{(x_0^2 + y_0^2 + z_0^2)^{3/2}}.$$

To find the field at a point P in terms of the coordinates of the moving frame, let $\mathbf{R}$ be the vector from q to P, so that $y_0 = y = R_y$ and $z_0 = z = R_z$ (since there is no motion in the y or z directions), and $x_0 = \gamma_0(x + v_0 t) = \gamma_0 R_x$ by the Lorentz transformation. Converting to cylindrical coordinates,

$$\begin{aligned} \mathbf{E} &= \frac{q}{4\pi\epsilon_0} \frac{\gamma_0 \mathbf{R}}{(\gamma_0^2 R^2 \cos^2\theta + R^2 \sin^2\theta)^{3/2}} \\ &= \frac{q}{4\pi\epsilon_0} \frac{1 - v_0^2/c^2}{[1 - (v_0^2/c^2)\sin^2\theta]^{3/2}} \frac{\hat{\mathbf{R}}}{R^2}. \end{aligned} \tag{9}$$

Using Eq. (9) we can check that Gauss's Law, $\int \mathbf{E}\cdot d\mathbf{a} = (1/\epsilon_0)Q_{enc}$, is obeyed by the field of a point charge in uniform motion, by integrating over a sphere of radius R centered on the charge (Problem 12.43a):

$$\int \mathbf{E}\cdot d\mathbf{a} = \int \left(\frac{1}{4\pi\epsilon_0}\frac{q(1-v_0^2/c^2)}{[1-(v_0^2/c^2)\sin^2\theta]^{3/2}}\frac{\hat{\mathbf{r}}}{R^2}\right)\cdot\left(R^2\sin\theta d\theta d\phi\hat{\mathbf{r}}\right)$$

$$= \frac{q}{2\epsilon_0}\left(1-\frac{v_0^2}{c^2}\right)\int_0^\pi \frac{\sin\theta}{[1-(v_0^2/c^2)\sin^2\theta]^{3/2}}d\theta = \frac{q}{2\epsilon_0}\left(1-\frac{v_0^2}{c^2}\right)\left(\frac{2}{1-v_0^2/c^2}\right) = \frac{q}{\epsilon_0}.$$

Once we have the electric field of a point charge in uniform motion, calculating the magnetic field is not too difficult. Since there exists a frame (the particle's rest frame) in which the magnetic field is zero, we can use Eq. (7) to write

$$\mathbf{B} = -\frac{1}{c^2}(\mathbf{v}\times\mathbf{E}) = -\frac{q\gamma_0}{c^2 4\pi\epsilon_0}\frac{\hat{\mathbf{x}}\times(R_x\hat{\mathbf{x}}+R_y\hat{\mathbf{y}}+R_x\hat{\mathbf{z}})}{(\gamma_0^2R_x^2+R_y^2+R_z^2)^{3/2}}$$

$$= \frac{q\mu_0}{4\pi}\frac{v(1-v^2/c^2)\sin\theta}{[1-(v^2/c^2)\sin^2\theta]^{3/2}}\frac{\hat{\boldsymbol{\phi}}}{R^2}.$$