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Physics 111

The Rings of Saturn

Mimas, the seventh of Saturn's known moons, causes perturbations in the structure of Saturn's rings. We will numerically examine the effect of Mimas on a small satellite orbiting Saturn through a technique called *mapping*. By approximating how the $(n+1)$th state depends on the nth state of a system, we are able to describe its progression.

The two forces we will consider acting on objects near Saturn are the gravitational pull of Saturn and of Mimas. We can account for the force due to Saturn using Kepler's Third Law, which relates the period of an orbiting object to the semimajor axis of the elliptic orbit:

$$\tau^2 = \frac{4\pi\mu}{k} a^3. \tag{1}$$

For an object of mass $m \ll M_S$, we approximate $\mu \approx m$. Since $k = GM_S m$, Eq. (1) becomes

$$\tau^2 = \frac{4\pi}{GM_S} a^3. \tag{2}$$

If we approximate the orbits of Mimas and our satellite as circles of radii σ and r, respectively, we can write the angular distance travelled by the satellite during one revolution of Mimas as

$$\Delta\theta = 2\pi \frac{\tau_M}{\tau_s} \approx 2\pi \left(\frac{4\pi/GM_S}{4\pi/GM_S} \frac{\sigma^3}{r^3} \right)^{\frac{1}{2}} = 2\pi \left(\frac{\sigma}{r} \right)^{\frac{3}{2}} \tag{3}$$

Letting θ be the relative angle between the satellite and Mimas, so that on the nth orbit the satellite is at a distance r_n from Saturn and an angle θ_n away from Mimas, we can now calculate the relative angle for the $(n+1)$th orbit:

$$\theta_{n+1} = \theta_n + 2\pi \left(\frac{\sigma}{r_n} \right)^{\frac{3}{2}}. \tag{4}$$

The other force to consider is the gravitational pull of Mimas. The radial acceleration experienced by the satellite can be expressed as

$$\frac{\Delta v_r}{\Delta t} \approx \frac{v_r(t+\Delta t) - v_r(t)}{\Delta t} \approx \frac{r(t+\Delta t) - r(t)}{(\Delta t)^2} - \frac{r(t) - r(t-\Delta t)}{(\Delta t)^2} = \frac{r(t+\Delta t) - 2r(t) + r(t-\Delta t)}{(\Delta t)^2}. \tag{5}$$

We can thus approximate

$$r_{n+1} - 2r_n + r_{n-1} = f(r_n, \theta_n). \tag{6}$$

If we make some large approximations by assuming a periodic inverse-square perturbing force that is periodic in θ_n, this leads to the radial equation:

$$r_{n+1} = 2r_n - r_{n+1} - a\frac{\cos\theta_n}{(r_n - \sigma)^2} \tag{7}$$

where $a = 2 \times 10^{12}$ km^3.

Using Eqs. (4) and (7), we wrote a program in IDL to investigate this two-dimensional map for r_n and θ_n. For initial r values between the radius of Saturn $R_S = 60.4 \times 10^3$ km and the radius of Mimas's orbit $\sigma = 185.7 \times 10^3$ km, we plotted points $(r_n \cos\theta_n, r_n \sin\theta_n)$, which is the position of the satellite with the xy-axis fixed relative to Mimas.

Figures (1) through (4) illustrate some of the results of our program. The initial conditions used to generate each graph are displayed at the top, with R in units of 10^3 km. Figures (1) and (2) show a stable orbit and a chaotic orbit. Figures (3) and (4) demonstrate the sensativity of this system to initial conditions; by changing the parameter a from 2000 to 2001, we eliminated the two attractors that are seen in Figure (3), resulting in a single stable orbit.