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Physics 116

Optical Solitons

An optical soliton is a pulse that travels without distortion due to dispersion or other effects. They are a nonlinear phenomenon caused by self-phase modulation (SPM), which means that the electric field of the wave changes the index of refraction seen by the wave (Kerr effect). SPM causes a red shift at the leading edge of the pulse. Solitons occur when this shift is canceled due to the blue shift at the leading edge of a pulse in a region of anomalous dispersion, resulting in a pulse that maintains its shape in both frequency and time. Solitons are therefore an important development in the field of optical communications.

Following closely from A. Ghatak and K. Thyagarajan's *Introduction to Fiber Optics*, we will consider the theoretical description of solitary waves.

First, let us consider the nonlinear effect of SPM. Recall from Hecht Ch. 13 that for nonlinear media, the electric polarization of a wave is described by

$$P = \epsilon_0 \chi E + \epsilon_0 \chi^{(2)} E^2 + \epsilon_0 \chi^{(3)} e^3 + \cdots , \tag{1}$$

where χ is the dielectric susceptibility. For an optical fiber, $\chi^{(2)} = 0$, so we keep only the first and third terms (also neglecting weaker higher-order terms). By considering the polarization of a plane wave, $E = E_0 \cos(\omega t - kz)$, whose intensity is $I = c\epsilon_0 n_0 E_0^2/2$, and remembering that the general relationship between polarization and refractive index is $P = \epsilon_0(n^2 - 1)E$, we find ... [Space for you to take notes!]

$$n = n_0 + n_2 I, \tag{2}$$

where

$$n_2 = \frac{3}{4}\frac{\chi^{(3)}}{c\epsilon_0 n_0^2}. \tag{3}$$

Now, let us consider a pulse in a nonlinear dispersive medium, which is approximately described by the so-called non-linear Schrödinger equation (NLSE),

$$-i\left(\frac{\partial f}{\partial z} + \frac{1}{v_g}\frac{\partial f}{\partial t}\right) - \frac{\alpha}{2}\frac{\partial^2 f}{\partial t^2} + \Gamma\left|f\right|^2 f = 0, \tag{4}$$

where

$$\frac{1}{v_g} = \left.\frac{dk}{d\omega}\right|_{\omega=\omega_0} \tag{5}$$

$$\alpha = \left.\frac{d^2 k}{d\omega^2}\right|_{\omega=\omega_0} \tag{6}$$

$$\Gamma = \frac{1}{2}\omega_0\epsilon_0 n_0 n_2 \tag{7}$$

and $f(z,t)$ is the envelope of the pulse:

$$E(z,t) = \exp\left[i\left(\omega_0 t - k_0 z\right)\right] f(z,t). \tag{8}$$

It is often easier to write Eq. (4) in a frame moving at the group velocity by making the substitution

$$T = t - \frac{z}{v_g}, \tag{9}$$

in which case Eq. (4) becomes

$$-i\frac{\partial f}{\partial z} - \frac{\alpha}{2}\frac{\partial^2 f}{\partial T^2} + \Gamma\left|f\right|^2 f = 0. \tag{10}$$

If we consider only the first term of Eq. (10), which describes a wave propagating through a nondispersive linear medium, we find that any pulse propagates without change.

If we consider the first two terms of Eq. (10), which describe a wave propagating through a dispersive linear medium, we find that

$$f(z,T) = \int A(\Omega) e^{i(\Omega T - \alpha\Omega^2 z/2)} d\Omega. \tag{11}$$

For a Gaussian input pulse,

$$E(z=0,t) = E_0 e^{-t/\tau_0^2} e^{i\omega_0 t}, \tag{12}$$

the solution is

$$f(z,t) = \left(1 + i\frac{2\alpha z}{\tau_0^2}\right)^{-1/2} \exp\left[-\frac{\left(t - \frac{z}{v_g}\right)^2}{(\tau_0^2 + 2i\alpha z)}\right]. \tag{13}$$

The pulse whose envelope is described by Eq. (13) is plotted at two different times in Figure (1) for a linear medium with anomalous dispersion. (In a region of normal dispersion, the chirping would occur in the opposite direction.)

If we consider only the first and the third terms of Eq. (10), which describe a wave propagating through a nondispersive nonlinear medium, we find that

$$f(z,T) = f_0(T) \exp\left[-i\Gamma \left|f_0(T)\right|^2 z\right]. \tag{14}$$

This solution assumes that the nonlinear effect is weak enough that v_g is intensity independent, which results in the fact that the envelope maintains its shape over time. The pulse whose envelope is described by Eq. (14) is illustrated in Figure (2) for the initial envelope f_0 of a Gaussian pulse.

Now, let us consider the soliton solution to Eq. (10). If we demand that the solution has the form $f(z,T) = E_0\psi(T)e^{-i\phi(z)}$, we end up with the envelope function

$$f(z,t) = E_0 \text{ sech } \left[\gamma\left(t - \frac{z}{v_g}\right)\right] e^{igz} \tag{15}$$

where $2g = -\alpha\gamma^2 = \Gamma E_0^2$. This means that in order for the input wave to propagate without change, it must have the form of a hyperbolic secant times a cosine. To see how this compares to a standard Gaussian pulse, I plotted both in Figure (3). Figure (4) shows the propagation of a soliton. As expected, it has the same form at a later time.

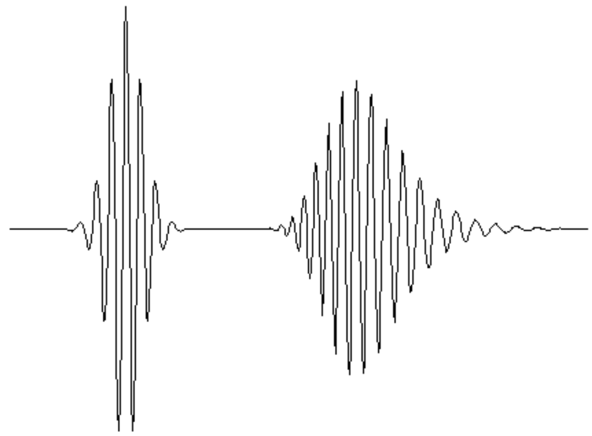

Figure 1: Gaussian wave propagating in the presence of dispersion only. Shown is the superposition of the pulse at two instances in time, plotted along the z-axis. The pulse undergoes broadening and chirping.

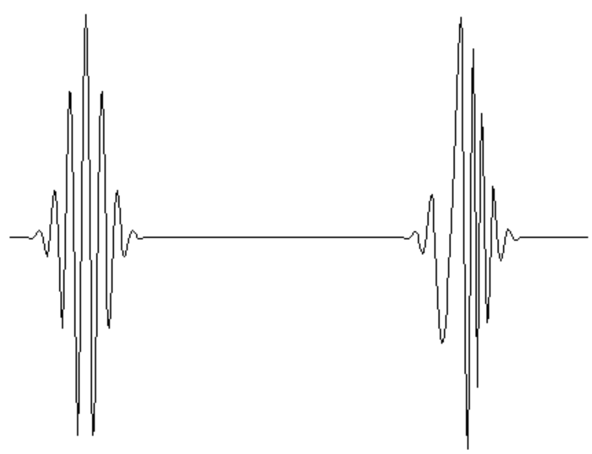

Figure 2: Gaussian wave propagating in the presence of SPM only. Note that the envelope remains the same, but there is a chirping effect.

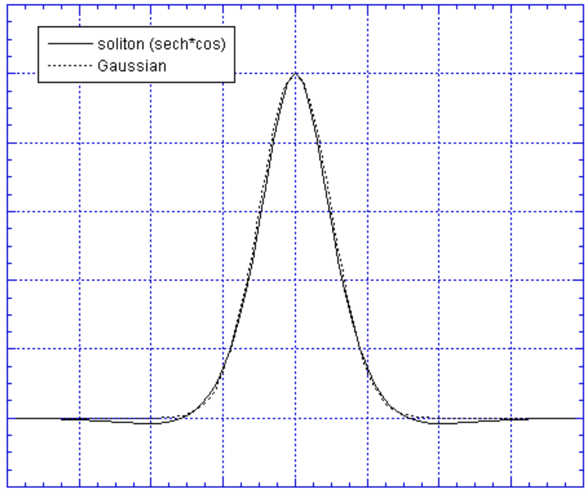

Figure 3: Envelope of soliton solution to NLSE compared with envelope of a Gaussian pulse.

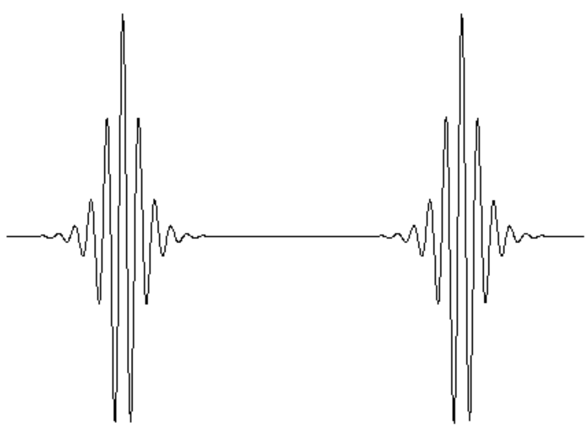

Figure 4: Solitary wave propagating through non-linear, dispersive medium. The pulse shape does not change with time.