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Physics 111

The Tippy-Top

General Description

As seen in Figure (1), when a tippy-top is spun on a smooth surface, it turns itself upside-down to rotate on its stem. In the process, it raises its center of mass and reverses its direction of rotation. Similarly, when a class ring is spun with the stone down, it lofts the stone in the air, and when a hard-boiled egg is spun flat, it rises to spin on one end.

This peculiar behavior has intrigued physicists since at least the 19th century, but it was not accurately described until the 1950's, and even then the description was limited by large approximations.

Friction is the key to understanding the behavior of the tippy-top. Since the CM is elevated during inversion (increasing the potential energy), the rotational kinetic energy must decrease. Thus, the total angular momentum $\mathbf{L}$ must decrease, which requires an external torque. $\mathbf{L}$ is almost entirely in the vertical $\hat{z}$ direction, so it cannot be changed by the normal or gravitational forces, since both point along the z axis. Friction is therefore necessary to explain the top's peculiar behavior.

A Simple Model[1]

Because the CM of a tippy-top is near the center of curvature (above which the top spins), as seen in Figure (2), the gravitational torque can be neglected. Note that the frictional force is coming out of the page, opposing the direction of slipping. The frictional torque is nearly horizontal and time averages to zero, so by neglecting it as well, we have no external torques on the top. Assuming that the tippy-top is initially spun vertically, this means that the angular momentum remains a constant along the z-axis.

Now let $\mathbf{L}$ be the angular momentum described in the body frame. Using the definition of torque and the general relation between the time derivative of a vector in the fixed and rotating frames, we have

$$\mathbf{N} = \frac{d\mathbf{L}_{\mathbf{fixed}}}{dt} = \frac{d\mathbf{L}}{dt} + \boldsymbol{\omega} \times \mathbf{L} \tag{1}$$

Since the tippy-top is approximately spherical, we let $I_1 \cong I_2 \cong I_3$, so that $\mathbf{L} \cong I\boldsymbol{\omega}$ and $\boldsymbol{\omega} \times \mathbf{L} \cong 0$. $\mathbf{L}$ remains vertical and $\mathbf{N}$ remains horizontal. Thus, by Eq. (1), $\frac{d\mathbf{L}}{dt}$ is always perpendicular to $\mathbf{L}$, causing $\mathbf{L}$ to precess uniformly in the body frame.

Taking the component of Eq. (1) along the symmetry axis of the top, $\hat{\mathbf{e}}_3$, we have

$$\hat{\mathbf{e}}_3 \cdot \mathbf{N} = \frac{d(\hat{\mathbf{e}}_3 \cdot \mathbf{L})}{dt} \tag{2}$$

Since $\hat{\mathbf{e}}_3 \cdot \mathbf{N} = N\cos(\theta + \frac{\pi}{2}) = -N\sin\theta$ and $\hat{\mathbf{e}}_3 \cdot \mathbf{L} = L\cos\theta$, Eq. (2) becomes

$$-N\sin\theta = \frac{d}{dt}L\cos\theta = -L\dot{\theta}\sin\theta + \dot{L}\cos\theta \tag{3}$$

We have assumed that $\dot{\mathbf{L}} = 0$, and so we have

$$\dot{\theta} = \frac{N}{L} \cong \frac{\mu M g R}{I\omega} \tag{4}$$

We see that θ increases, causing the top to begin to flip.

[1] Barger and Olsson, *Classical Mechanics: A Modern Perspective*, 1973.

Once the stem of the top touches the table, we have an ordinary rising top. **N** will still be perpendicular to **L**, and **L** will still change in the direction of **N**. However, since we now assume that **L** lies along the symmetry axis $\hat{\mathbf{e}}_3$, $\dot{\theta}$ is negative, and the tippy-top rights itself to the stable verticle position, where it will remain as long as

$$\omega_3 \geq \frac{(4MgRI)^{\frac{1}{2}}}{I_3} \tag{5}$$

as we saw in problem A1.

Other Considerations

While this analysis has given us a basic understanding of how the tippy-top works, we have made some large approximations. **L** is not really a constant during the first part of inversion, then suddenly switching from being along the fixed vertical axis to the body axis when the stem touches the table. Also, we assumed that the center of mass was on the axis of rotation, which allowed us to ignore the normal and gravitational forces.

A more rigorous analysis leads to six nonlinear differential equations (three translational and three angular), and a numerical solution allows you to solve for the motion of the top. The θ for a small coefficient of friction is seen in Figure (3).[2] Note that it is not monotonically increasing, but is nutating as the top flips.

[2] R.J. Cohen, *Am. J. Phys.* 45, 12 (1977).